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Topological and Borel compactifications of Polish G-spaces

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Abstract

We investigate the relationships between topological and Borel G-spaces, where G is a Polish group. We show that every Polish G-space can be topologically and equivariantly embedded into a compact Polish G-space iff G is locally compact. This answers a question of Kechris. It also provides a striking contrast to the recent result of Becker and Kechris which states that every *Borel* G-space can be Borel-embedded into a compact Polish G-space. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

An *action* of a group G (with neutral element e) on a set X is a function $\alpha : G \times X \to X$ such that $\alpha(e, x) = x$ for all $x \in X$ and $\alpha(g_1 g_2, x) = \alpha(g_1, \alpha(g_2, x))$ for all $g_1, g_2 \in G$ and $x \in X$.

In topological dynamics, for example, G is a topological group, X is a topological space, and α is *continuous*. The triple $\langle G, X, \alpha\rangle$ is called a *G-space*, or a *topological G-space*. For basic information on topological G-spaces, see [20,25].

In descriptive set theory, one usually requires G to be a Polish (= complete, metrizable and separable) group and X a *standard Borel space*, that is, a measurable space (a set X and a σ-algebra $\mathcal{S}$ of subsets of X) such that there exists a Polish topology on X with $\mathcal{S}$ its σ-algebra of Borel sets. The assumption on the action α is weakened from continuity to Borel measurability. If X and Y are separable metrizable spaces, then a function $f : X \to Y$ is *Borel measurable* if $f^{-1}(B)$ is a Borel set in X for every Borel set B in Y. Thus, a *Borel*

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G-space is a triple $\langle G, X, \alpha\rangle$, where G is a *Polish* group, X is a standard Borel space, and $\alpha : G \times X \to X$ is a Borel-measurable action. See [3] for more on Borel G-spaces.

It is natural to compare topological and Borel G-spaces. What (kinds of) theorems are true in both contexts? How much flexibility is gained by relaxing the continuity of the action to Borel-measurability?

We have found a striking contrast in the area of G-space *compactifications*. A *G-compactification* of a topological G-space $\langle G, X, \alpha\rangle$ consists of a topological G-space $\langle G, cX, \alpha'\rangle$, where $\varphi : X \to cX$ is a compactification of X and $\alpha'|_{G\times X} = \alpha$. In other words, X is topologically and *equivariantly* embedded in cX, that is, $\varphi(gx) = g\varphi(x)$ for all $g \in G$ and $x \in X$.

A G-space is called *G-Tychonoff* if it has a G-compactification. For example, it is well known (see, for example, [4] and [17]) that for *every* topological group G, the G-space $\langle G, G, \alpha_L\rangle$ ($\alpha_L(g_1, g_2) = g_1g_2$) is G-Tychonoff. In fact, $\langle G, G, \alpha_L\rangle$ has a *maximal* G-compactification, denoted by $\beta_G G$ and called the *greatest ambit*. Jan de Vries [21] proved that every coset G-space $\langle G, G/H, \alpha_L^*\rangle$ is G-Tychonoff, where $\alpha_L^*(g_1, g_2H) = (g_1g_2)H$. This includes the previous result as the particular case $H = \{e\}$. Ludescher and de Vries [10] showed that any G-space under an *equicontinuous* action is G-Tychonoff.

G is a *V-group* if every Tychonoff G-space is G-Tychonoff. In [21], Jan de Vries posed the "compactification problem" in its full generality: is every topological group G a V-group? Carlson [5] had earlier given a positive answer for the case $G = \mathbb{R}$. Palais [16] showed that every compact Lie group is a V-group. De Vries [22] and Antonyan [1] independently proved that every compact group is a V-group. In [24], de Vries improved these results.

Theorem 1.1 (de Vries). *Every locally compact group is a V-group.*

The converse of Theorem 1.1 is still open, and, in fact, it is not known whether the additive group $\mathbb{Q}$ of rational numbers is a V-group. However, in [15] Megrelishvili and the present author showed that the class of non-V-groups is, in fact, rather large and contains all groups which are $\aleph_0$-bounded (for example, second countable) and not locally precompact. Recall that a topological group G is called *$\aleph_0$-bounded* [2,6] if for every neighborhood V of e there exists a countable subset S of G such that $SV = G$. Guran [6] proved that G is $\aleph_0$-bounded iff G is a topological subgroup of a product of second countable topological groups. If G is separable, Lindelöf, or satisfies the countable chain condition, then G is $\aleph_0$-bounded.

G is *locally precompact* if it is a subgroup of a locally compact group, or, equivalently, if its sup-completion (the completion with respect to its two-sided uniformity) is locally compact.

Theorem 1.2 [15]. *If G is an $\aleph_0$-bounded topological group which is not locally precompact, then G is not a V-group.*

If G is *Polish*, we have a complete answer to the compactification problem.

Theorem 1.3 [15]. *Let G be a Polish group. Then G is a V-group iff G is locally compact.*

There is also much recent work in the area of *universal* G-spaces. In [7], Hjorth proves the existence of a *continuously universal* Polish G-space for every Polish group G. Recall also that for every second countable G and second countable G-Tychonoff X, there exists an equivariant topological embedding of the pair (G, X) into the pair $(H(I^{\aleph_0}), I^{\aleph_0})$, where $I^{\aleph_0}$ denotes the Hilbert cube and $H(I^{\aleph_0})$ denotes the topological group of all homeomorphisms on the Hilbert cube. This follows as a particular case of Theorem 2.7 from [14].

Turning to Borel G-spaces, we find the notion of *universal Borel* G-spaces. Fix a Polish group G. A *universal* Borel G-space is a Borel G-space $\mathcal{U}_G$ such that every Borel G-space can be Borel-embedded into $\mathcal{U}_G$. Mackey [11] and Varadarajan [19] proved that if G is Polish and *locally compact*, then there exists a universal Borel G-space. Becker and Kechris [3] have recently extended this result to *all* Polish groups, even ones which are *not* locally compact. In fact, they have gone even one step further.

Theorem 1.4 [3, Theorem 2.6.6]. *For any Polish group G, there is a universal Borel G-space which is moreover a compact Polish G-space.*

This last result means that for *every* Polish group G, every Borel G-space can be Borel-embedded into a compact Polish G-space.

Our Main Theorem, which was conjectured by Kechris [9], shows that the situation is strikingly different for *topological* G-spaces.

Main Theorem. *Let G be a Polish group. Then every Polish G-space can be* (*topologically*) *embedded into a compact Polish G-space iff G is locally compact.*

Our proof of the above theorem uses the *strong Choquet game* from descriptive set theory. We also need the following result.

Theorem 1.5 [13]. *Every G-Tychonoff space X has a G-compactification Y such that $w(Y) \leqslant w(X) \cdot w(G)$ and* $\dim Y \leqslant \dim \beta_G X$.

2. Proof of the Main Theorem

We turn now to the proof of the Main Theorem. We need a little background from descriptive set theory. See, for example, [8].

Fact 2.1. *A locally compact Hausdorff space is Polish iff it is second countable.*

We will use the following game.

Definition 2.2. Given a nonempty topological space X, the *strong Choquet game* G_X^s between two players is defined as follows:

$$\begin{array}{llll} \text{Player I plays} & x_0, U_0 & & x_1, U_1 & \\ & & & & \cdots \\ \text{Player II plays} & & V_0 & & V_1 \end{array}$$

Players I and II take turns playing nonempty open subsets of X such that $U_0 \supseteq V_0 \supseteq U_1 \supseteq V_1 \supseteq \cdots$. Additionally, player I is required to play a point $x_n \in U_n$ and then player II must play $V_n \subseteq U_n$ with $x_n \in V_n$.

Player I wins the game if $\bigcap_n V_n = \emptyset$.

Player II wins the game if $\bigcap_n V_n (= \bigcap_n U_n) \neq \emptyset$.

A nonempty space X is called a *strong Choquet space* if player II has a winning strategy for the game G_X^s.

Theorem 2.3 (Choquet). *A nonempty, second countable topological space is Polish iff it is T_1, regular, and strong Choquet.*

We can now prove the Main Theorem.

Main Theorem 2.4. *Let G be a Polish group. Then every Polish G-space can be (topologically) embedded into a compact Polish G-space iff G is locally compact.*

Proof. ($\Leftarrow$) Let X be a Polish G-space. Since G is locally compact, X has a G-compactification, by Theorem 1.1. Since G and X are both second countable, there exists a second countable G-compactification Y of X, by Theorem 1.5. By Fact 2.1, Y is Polish.

($\Rightarrow$) Suppose G is Polish and not locally compact. Then G is second countable and not locally precompact. We will construct a Polish G-space which is not G-Tychonoff. This construction can also be found in [15].

Since G is not locally precompact, G does not act locally uniformly equicontinuously on $X = \beta_G G$ (see [15, Lemma 2.7]). We will first construct a G-space X_U for every U in a collection $\mathbb{B}$ (of cardinality $\chi(G)$) of basic neighborhoods of e in G. By [15, Fact 2.8], X is a G-limit of an inverse G-system of compact metrizable G-spaces X_i $(i \in I)$. Let μ and μ_i denote the unique compatible uniformity on X and X_i, respectively.

Let $U \in \mathbb{B}$. Since U does not act μ-uniformly equicontinuously on $\beta_G G$, there exists an index $i \in I$ such that U does not act μ_i-uniformly equicontinuously on (X_i, μ_i). Therefore, there is $\varepsilon \in \mu_i$ such that for every $\delta \in \mu_i$ there exist $(x_\delta, y_\delta) \in \delta$ and $g_\delta \in U$ such that

$$(g_\delta x_\delta, g_\delta y_\delta) \notin \varepsilon. \tag{$*$}$$

Thus we obtain nets $\langle x_\delta \rangle$, $\langle y_\delta \rangle$, $\langle g_\delta x_\delta \rangle$, and $\langle g_\delta y_\delta \rangle$ in (X_i, μ_i), indexed by the elements δ of μ_i. Passing to subnets if necessary, we may assume that there exist $x^U, a^U, b^U \in X$ such that $x_\delta \to x^U$, $y_\delta \to x^U$, $g_\delta x_\delta \to a^U$, and $g_\delta y_\delta \to b^U$. By $(*)$ we have $a^U \neq b^U$. Hence $(x^U, x^U) \in \Delta$, and $(a^U, b^U) \notin \Delta$, where $\Delta = \{(x, x) \mid x \in X_i\}$.

We put the "two-coordinate" action on the G-space $(X_i, \mu_i) \times (X_i, \mu_i)$ and then form the quotient G-space $Y_i = (X_i \times X_i)/\Delta$. Consider the quotient G-map $p: X_i \times X_i \to Y_i$. Let $z := p(x^U, x^U)$.

Next, let

$$X_U = (Y_i \times Y_i) \setminus \{(z,z)\}, \quad C^U = (\{z\} \times Y_i) \setminus \{(z,z)\},$$

and

$$D^U = (Y_i \times \{z\}) \setminus \{(z,z)\}.$$

Define the "one-coordinate" action $\alpha': G \times X_U \to X_U$ by

$$\alpha'(g,(x,y)) = (\alpha(g,x), y),$$

where α is the action of G on Y_i. This completes the construction of X_U from U.

Now form the topological G-sum $S = \bigoplus\{X_U\colon U \in \mathbb{B}\}$. Let $\alpha^*: G \times S \to S$ be the natural action. Define

$$C = \bigcup_{U \in \mathbb{B}} C^U, \qquad D = \bigcup_{U \in \mathbb{B}} D^U.$$

Finally, from $\langle G, S, \alpha^*\rangle$, we will construct a normal G-space $\langle G, S^+, (\alpha^*)^+\rangle$ which is not G-Tychonoff.

Let $\omega = \mathbb{N} \cup \{0\}$ carry the discrete topology. Let Y be the quotient space formed from $S \times \omega$ by identifying the pairs $(c, 2i+1)$ and $(c, 2i+2)$ for $c \in C$, $i \in \omega$, and the pairs $(d, 2i)$ and $(d, 2i+1)$ for $d \in D$, $i \in \omega$. Let $p: S \times \omega \to Y$ be the quotient map. For $n \in \omega$, let $i_n: S \to S \times \omega$ be the canonical injection $x \mapsto (x, n)$.

Fix a point $a \notin Y$, and let $S^+ = Y \cup \{a\}$. Topologize S^+ by setting Y to be an open subset with its quotient topology and the nth basic neighborhood of a to be

$$N_n(a) = \{a\} \cup p(i_{2n}(S \setminus C)) \cup \bigcup \{p(i_m(S))\colon m > 2n\}.$$

We now define $\alpha^+: G \times S^+ \to S^+$. For any $g \in G$, set $\alpha^+(g, a) = a$. For $p((x,n)) \in Y$, set $\alpha^+(g, p((x,n))) = p(i_n(\alpha^*(g,x)))$. Then $\langle G, S^+, \alpha^+\rangle$ is a normal G-space which is not G-Tychonoff (see [15, Theorem 3.2]).

From the construction of S and S^+, it is clear that if G is second countable, then S^+ can be chosen to be second countable. We will show that S^+ is, in fact, a Polish space. Since S^+ is normal, it is enough, by Theorem 2.3, to show that S^+ is strong Choquet.

Let $Y = S^+ \setminus \{a\}$. Then Y is locally compact, Hausdorff, and second countable. By Fact 2.1, Y is Polish. So Y is strong Choquet, by Theorem 2.3. Now we can define a winning strategy for player II in the game $G^s_{S^+}$.

Suppose player I plays x_0, U_0. If $a \notin U_0$, then U_0 is an open subset of Y. So player II can now apply his winning strategy for the game G^s_Y.

Otherwise, $a \in U_0$. If $x_0 \neq a$, then player II can play $U_0 \cap Y$. This will force player I to play x_1, U_1, where U_1 is an open subset of Y. So player II can now apply his winning strategy for G^s_Y.

Finally, suppose $x_0 = a$. Player II can play $V_0 = N_n(a)$, where $n \in \mathbb{N}$ is large enough so that $N_n(a) \subseteq U_0$. From this point on, if player I plays x_k, U_k, with $U_k \subseteq Y$, then player II applies his winning strategy for G^s_Y. Otherwise, player I always plays a, U_k. In this case, player II can play $V_k = N_{n_k}(a)$ for sufficiently large and increasing values of n_k. In the end, we will have

$$\bigcap_k V_k = \{a\},$$

and player II wins the game $G^s_{S^+}$.

Therefore S^+ is strong Choquet and, hence, Polish. □

3. Question

When can a Tychonoff G-space be topologically embedded into a compact *Borel* G-space?

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References

[1] S. Antonyan, Yu. Smirnov, Universal objects and bicompact extensions for topological groups of transformations, Soviet Math. Dokl. 23 (1981) 279–284.
[2] A.V. Arhangel'skiĭ, Classes of topological groups, Russian Math. Surveys 36 (1981) 521–526.
[3] H. Becker, A. Kechris, The Descriptive Set Theory of Polish Group Actions, Cambridge Univ. Press, Cambridge, 1996.
[4] R.B. Brook, A construction of the greatest ambit, Math. Systems Theory 4 (1970) 243–248.
[5] D.H. Carlson, Extensions of dynamical systems via prolongations, Funkcial. Ekvac. 14 (1971) 35–46.
[6] I.I. Guran, On topological groups close to being Lindelöf, Soviet Math. Dokl. 23 (1981) 173–175.
[7] G. Hjorth, A universal Polish G-space, Topology Appl. 91 (1999) 141–150.
[8] A. Kechris, Classical Descriptive Set Theory, Springer, New York, 1995.
[9] A. Kechris, e-mail communication, October, 1997.
[10] H. Ludescher, J. de Vries, A sufficient condition for the existence of a G-compactification, Nederl. Akad. Wetensch. Proc. Ser. A 83 (1980) 263–268.
[11] G.W. Mackey, Point realizations of transformation groups, Illinois J. Math. 6 (1962) 327–335.
[12] M.G. Megrelishvili, Equivariant completions and compact extensions, Bull. Acad. Sci. Georgian SSR 115 (1984) 21–24.
[13] M. Megrelishvili, Compactification and factorization in the category of G-spaces, in: J. Adámek, S. MacLane (Eds.), Categorical Topology and its Relation to Analysis, Algebra and Combinatorics, World Scientific, Singapore, 1989, pp. 220–237.
[14] M. Megrelishvili, Free topological G-groups, New Zealand J. Math. 25 (1996) 59–72.
[15] M. Megrelishvili, T. Scarr, Constructing Tychonoff G-spaces which are not G-Tychonoff, Topology Appl. 86 (1998) 69–81.

[16] R. Palais, The classification of G-spaces, Mem. Amer. Math. Soc. 36 (1960) 25.
[17] S. Teleman, Sur la représentation linéaire des groupes topologiques, Ann. Sci. Ecole Norm. Sup. 74 (1957) 319–339.
[18] I. Tree, Constructing regular spaces that are not completely regular, Houston J. Math. 21 (3) (1995) 613–622.
[19] V.S. Varadarajan, Groups of automorphisms of Borel spaces, Trans. Amer. Math. Soc. 109 (1963) 191–220.
[20] J. de Vries, Topological Transformation Groups I, Math. Centre Tracts 65, Mathematisch Centrum, Amsterdam, 1975.
[21] J. de Vries, Can every Tychonoff G-space equivariantly be embedded in a compact Hausdorff G-space?, Math. Centrum, Amsterdam, Afd. Zuivere Wisk. 36 (1975).
[22] J. de Vries, A note on compactifications of G-spaces, Math. Centrum, Amsterdam, Afd. Zuivere Wisk. 61 (1976).
[23] J. de Vries, Equivariant embeddings of G-spaces, in: General Topology and its Relations to Modern Analysis and Algebra IV, Part B, Proc. 4th Prague Topological Symposium (1976), Prague (1977) 485–493.
[24] J. de Vries, On the existence of G-compactifications, Bull. Acad. Polon. Sci. Ser. Math. 26 (1978) 275–280.
[25] J. de Vries, Elements of Topological Dynamics, Kluwer Academic Publishers, Dordrecht, 1993.